Exam Statistical Methods in Physics

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Before you start, read the following:

- Write your name and student number on top of your exam;
- Illegible writing will be graded as incorrect;
- All problems have equal weight;
- Annexes:
  - Integral of the Standard Normal distribution
  - Quantiles of the Chi-squared distribution
  - Quantiles of Student's t-distribution
- *Good luck!*

$t = \frac{\sqrt{N}(\bar{x}-\mu)}{s}$ comp th with exp.

$\mu = 0$

$v = \frac{N}{N-2}$

Problem 1

Read the following statements carefully, and indicate if they are true or false:

(a) If F is a cumulative distribution function of x, $F(x-dx) > F(x)$ for $dx > 0$;

(b) $E[A+B] = E[A] + E[B]$;

(c) For a strictly monotonically decreasing transformation $Y = h(X)$, the PDF of Y is given by $g(Y) = -f(X)/h'(X)$ if the PDF of X is $f(X)$;

(d) The distribution $f(x) = \frac{1/\pi}{x^2-6x+10}$ has variance $\frac{6}{8}\sqrt{\pi}$;

(e) If the joint PDF of x and y is $f(x,y) = \frac{1}{\pi}e^{-\frac{1}{2}r^2}$, with $r = \sqrt{x^2+y^2}$, then x and y are independent;

(f) The multivariate Normal distribution is fully specified by the covariance matrix;

(g) If P and Q are independently χ^2 distributed with N and M degrees of freedom, then $Z = P + Q$ is χ^2 distributed with $N + M$ degrees of freedom;

(h) If $x_1, x_2, \cdots x_N$ are independent and normal distributed with the same μ and σ, with the sample mean m and the sample variance s then $t = \frac{\sqrt{N}(m-\mu)}{s}$ has a Student-t distribution with $N-1$ degrees of freedom;

(i) The probability that a consistent estimator deviates more an arbitrarily small amount from the truth can be made arbitrarily small;

(j) A consistent estimator cannot be biased;

(k) An estimator $\hat{\theta}$ is unbaised if $b_N(\hat{\theta}) = E[\hat{\theta}_N - \theta_0] = 0 \;\; \forall \; N$;

(l) A 95% probability interval is uniquely determined;

(m) If the test statistic falls in the critical region, you accept the null hypothesis;

(n) In hypothesis testing, a type-I error leads to inefficiency;

(o) For a fixed loss in testing a hypothesis, the contamination can be made arbitrarily small.

Problem 2

You are building an X-ray telescope, that consist of an 10×10 array of (circular) detectors (see figure). When hit, these detectors detect incoming X-ray photons with an efficiency of 10%. Each detector has an independent channel in the readout electronics. At the beginning of a measurement, the readout is activated and all channels are put in the "not-hit" state. If the attached detector detects a photon, the recording electronics will switch from the "not-hit" state into the "hit" state (or stay in the "hit" state if it was already there). At the end of a measument, the number of detectors in the "hit" state, k, is counted, and stored.

(a) What is the distribution of the number of X-rays that are *detected* in a single detector, if a large number of X-rays, $N \gg 1$, enter the telescope during the measurement?

(b) What is the probability that a detector is "hit" at the end of the measurement?

(c) What is the distribution of the number of detectors that are "hit"?

(d) For which N are all detectors "hit" with 95% probability?

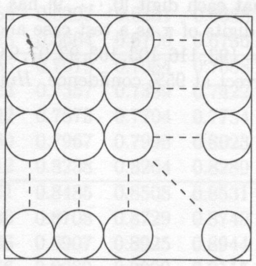

Figure 1: Layout of the X-ray detector for Problem 2

Problem 3

A very inexperienced archer shoots n arrows at a disc of (unknown) radius ρ. The disc is hit every time, but at completely random places (x_i, y_i). Determine the maximum likelihood estimate for ρ.

Problem 4

Consider the following data set for an experiment in which the observable t is measured, with $t \in [0, 10]$. The measurement is repeated ten times and yields $t = \{$ 6.4, 4.3, 5.5, 7.1, 6.4, 6.8, 0.9, 1.2, 0.5, 8.2 $\}$. You expect the PDF of t to be

$$t : f(t) = \mathcal{N} e^{-t^2/8}.$$

However, you are not sure whether the experiment was working properly. It might be that you measured noise, in which case t would be uniformly distributed.

Test at 90% significance whether the experiment worked properly. *Hint: consider the Neyman-Pearson procedure and the symmetry of the Normal distribution.*

Problem 5

A mathematician claims that irrational numbers cannot be represented by a repeating sequence of decimals and that each digit $[0, \cdots, 9]$ has an equal probability of showing up. You take the first 1000 digits of π as a test case and count the frequency of the occurence of the digits as $f[i] = \{93, 116, 103, 103, 93, 97, 94, 95, 101, 105\}$. Test whether the mathematician's claim is correct at 95% confidence. *Hint: you may ignore correlations.*

Table 1: Integral of the Standard Normal distribution: $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$.

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.00	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.10	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.20	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.30	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.40	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.50	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.60	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.70	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.80	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.90	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.00	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.10	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.20	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.30	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.40	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.50	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.60	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.70	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.80	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.90	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.00	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.10	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.20	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.30	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.40	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.50	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.60	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.70	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.80	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.90	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.00	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.10	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.20	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.30	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.40	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.50	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.60	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.70	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.80	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.90	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 2: Quantiles of the Chi-squared distribution: $\int_0^x \chi^2(NDF)dx = \alpha$

NDF; α	0.005	0.010	0.025	0.050	0.900	0.950	0.975	0.990	0.995
1	0.000	0.000	0.001	0.004	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	33.196	36.415	39.364	42.980	45.559
25	10.520	11.524	13.120	14.611	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	36.741	40.113	43.195	46.963	49.645
28	12.461	13.565	15.308	16.928	37.916	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	39.087	42.557	45.722	49.588	52.336
30	13.787	14.953	16.791	18.493	40.256	43.773	46.979	50.892	53.672
40	20.707	22.164	24.433	26.509	51.805	55.758	59.342	63.691	66.766
50	27.991	29.707	32.357	34.764	63.167	67.505	71.420	76.154	79.490
60	35.534	37.485	40.482	43.188	74.397	79.082	83.298	88.379	91.952
70	43.275	45.442	48.758	51.739	85.527	90.531	95.023	100.425	104.215
80	51.172	53.540	57.153	60.391	96.578	101.879	106.629	112.329	116.321
90	59.196	61.754	65.647	69.126	107.565	113.145	118.136	124.116	128.299
100	67.328	70.065	74.222	77.929	118.498	124.342	129.561	135.807	140.169

Table 3: Quantiles of Student's t-distribution: $\int_{-\infty}^{x} Stud(NDF)dx = \alpha$

NDF; α	0.550	0.600	0.680	0.750	0.900	0.950	0.975	0.990	0.995
1	0.158	0.325	0.635	1.000	3.078	6.314	12.706	31.821	63.657
2	0.142	0.289	0.546	0.816	1.886	2.920	4.303	6.965	9.925
3	0.137	0.277	0.518	0.765	1.638	2.353	3.182	4.541	5.841
4	0.134	0.271	0.505	0.741	1.533	2.132	2.776	3.747	4.604
5	0.132	0.267	0.497	0.727	1.476	2.015	2.571	3.365	4.032
6	0.131	0.265	0.492	0.718	1.440	1.943	2.447	3.143	3.707
7	0.130	0.263	0.489	0.711	1.415	1.895	2.365	2.998	3.499
8	0.130	0.262	0.486	0.706	1.397	1.860	2.306	2.896	3.355
9	0.129	0.261	0.484	0.703	1.383	1.833	2.262	2.821	3.250
10	0.129	0.260	0.482	0.700	1.372	1.812	2.228	2.764	3.169
11	0.129	0.260	0.481	0.697	1.363	1.796	2.201	2.718	3.106
12	0.128	0.259	0.480	0.695	1.356	1.782	2.179	2.681	3.055
13	0.128	0.259	0.479	0.694	1.350	1.771	2.160	2.650	3.012
14	0.128	0.258	0.478	0.692	1.345	1.761	2.145	2.624	2.977
15	0.128	0.258	0.477	0.691	1.341	1.753	2.131	2.602	2.947
16	0.128	0.258	0.477	0.690	1.337	1.746	2.120	2.583	2.921
17	0.128	0.257	0.476	0.689	1.333	1.740	2.110	2.567	2.898
18	0.127	0.257	0.476	0.688	1.330	1.734	2.101	2.552	2.878
19	0.127	0.257	0.475	0.688	1.328	1.729	2.093	2.539	2.861
20	0.127	0.257	0.475	0.687	1.325	1.725	2.086	2.528	2.845
21	0.127	0.257	0.475	0.686	1.323	1.721	2.080	2.518	2.831
22	0.127	0.256	0.474	0.686	1.321	1.717	2.074	2.508	2.819
23	0.127	0.256	0.474	0.685	1.319	1.714	2.069	2.500	2.807
24	0.127	0.256	0.474	0.685	1.318	1.711	2.064	2.492	2.797
25	0.127	0.256	0.473	0.684	1.316	1.708	2.060	2.485	2.787
26	0.127	0.256	0.473	0.684	1.315	1.706	2.056	2.479	2.779
27	0.127	0.256	0.473	0.684	1.314	1.703	2.052	2.473	2.771
28	0.127	0.256	0.473	0.683	1.313	1.701	2.048	2.467	2.763
29	0.127	0.256	0.473	0.683	1.311	1.699	2.045	2.462	2.756
30	0.127	0.256	0.472	0.683	1.310	1.697	2.042	2.457	2.750
40	0.126	0.255	0.471	0.681	1.303	1.684	2.021	2.423	2.704
50	0.126	0.255	0.471	0.679	1.299	1.676	2.009	2.403	2.678
60	0.126	0.254	0.470	0.679	1.296	1.671	2.000	2.390	2.660
70	0.126	0.254	0.470	0.678	1.294	1.667	1.994	2.381	2.648
80	0.126	0.254	0.469	0.678	1.292	1.664	1.990	2.374	2.639
90	0.126	0.254	0.469	0.677	1.291	1.662	1.987	2.368	2.632
100	0.126	0.254	0.469	0.677	1.290	1.660	1.984	2.364	2.626
∞	0.126	0.253	0.468	0.674	1.282	1.645	1.960	2.326	2.576